

Quaternion algebras and the generalized Fibonacci-Lucas quaternions

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Abstract. In this paper, we introduce the generalized Fibonacci-Lucas quaternions and we prove that the set of these elements is an order—in the sense of ring theory— of a quaternion algebra. Moreover, we investigate some properties of these elements.

Key Words: quaternion algebras; Fibonacci numbers; Lucas numbers.

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1. Preliminaries

Let $(f_n)_{n \geq 0}$ be the Fibonacci sequence:

$$f_0 = 0; f_1 = 1; f_n = f_{n-1} + f_{n-2}, \quad n \geq 2$$

and $(l_n)_{n \geq 0}$ be the Lucas sequence:

$$l_0 = 2; l_1 = 1; l_n = l_{n-1} + l_{n-2}, \quad n \geq 2$$

In the paper [Ho; 61], A. F. Horadam generalized Fibonacci numbers in the following way:

$$h_0 = p, h_1 = q, h_n = h_{n-1} + h_{n-2}, \quad n \geq 2,$$

where p and q are arbitrary integer numbers.

In the paper [Ho; 63], A. F. Horadam introduced the Fibonacci quaternions and generalized Fibonacci quaternions. In their work [Fl, Sh; 12], C. Flaut and V. Shpakivskyi and later in the paper [Ak, Ko, To; 14], M. Akyigit, H. H Kosal, M. Tosun found some properties of the generalized Fibonacci quaternions. In the papers [Fl, Sa; 14] and [Fl, Sa, Io; 13], the authors gave the definitions of the Fibonacci symbol elements and Lucas symbol elements and they determined many properties of these elements. All these elements determine a \mathbb{Z} -module structure.

In this paper, we define the generalized Fibonacci-Lucas numbers and the generalized Fibonacci-Lucas quaternions and we prove that, in the second case, they determine an order of a quaternion algebra.

For other results regarding above notions, the reader is referred to [Fl; 06], [Sa, Fl, Ci; 09], [Fl, Sh; 13(1)], [Fl, Sa; 14], [Fl, Sa; 15], [Fl, St; 09].

2. Properties of the Fibonacci and Lucas numbers

From [Fib.], the following properties of Fibonacci numbers are known:

Proposition 2.1. *Let $(f_n)_{n \geq 0}$ be the Fibonacci sequence and let $(l_n)_{n \geq 0}$ be the Lucas sequence. Therefore the following properties hold:*

- i)
$$f_n^2 + f_{n+1}^2 = f_{2n+1}, \forall n \in \mathbb{N};$$
- ii)
$$f_{n+1}^2 - f_{n-1}^2 = f_{2n}, \forall n \in \mathbb{N}^*;$$
- iii)
$$l_n^2 - f_n^2 = 4f_{n-1}f_{n+1}, \forall n \in \mathbb{N}^*;$$
- iv)
$$l_n^2 + l_{n+1}^2 = 5f_{2n+1}, \forall n \in \mathbb{N};$$
- v)
$$l_n^2 = l_{2n} + 2(-1)^n, \forall n \in \mathbb{N}^*;$$
- vi)
$$f_{n+1} + f_{n-1} = l_n, \forall n \in \mathbb{N}^*;$$
- vii)
$$l_n + l_{n+2} = 5f_{n+1}, \forall n \in \mathbb{N};$$
- viii)
$$f_n + f_{n+4} = 3f_{n+2}, \forall n \in \mathbb{N};$$
- ix)
$$f_m l_{m+p} = f_{2m+p} + (-1)^{m+1} \cdot f_p, \forall m, p \in \mathbb{N};$$
- x)
$$f_{m+p} l_m = f_{2m+p} + (-1)^m \cdot f_p, \forall m, p \in \mathbb{N};$$
- xi)
$$f_m f_{m+p} = \frac{1}{5} \left(l_{2m+p} + (-1)^{m+1} \cdot l_p \right), \forall m, p \in \mathbb{N};$$
- xii)
$$l_m l_p + 5f_m f_p = 2l_{m+p}, \forall m, p \in \mathbb{N}.$$
- xiii)
$$f_n \text{ are even numbers if and only if } n \equiv 0 \pmod{3}.$$

In the next proposition, we will give other elementary properties of the Fibonacci and Lucas numbers. These properties will be necessary in the following.

Proposition 2.2. *Let $(f_n)_{n \geq 0}$ be the Fibonacci sequence and $(l_n)_{n \geq 0}$ be the Lucas sequence. Then we have that*

$$l_m l_{m+p} = l_{2m+p} + (-1)^m \cdot l_p, \forall m, p \in \mathbb{N}.$$

Proof. If we denote $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, using Binet's formula, we have $f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$, $\forall n \in \mathbb{N}$ and $l_n = \alpha^n + \beta^n$, $\forall n \in \mathbb{N}$.

i) Let $m, p \in \mathbb{N}$, $p \leq m$, therefore we have

$$\begin{aligned} l_m l_p - 5f_m f_p &= (\alpha^m + \beta^m) \cdot (\alpha^p + \beta^p) - (\alpha^m - \beta^m) \cdot (\alpha^p - \beta^p) = \\ &= 2\alpha^p \beta^p (\alpha^{m-p} + \beta^{m-p}) = 2(-1)^p \cdot l_{m-p}. \end{aligned}$$

It results $l_m l_p - 5f_m f_p = 2(-1)^p \cdot l_{m-p}$. Adding this equality with the equality from Proposition 2.1 (xii), we obtain $l_m l_p = l_{m+p} + (-1)^p \cdot l_{m-p}$, $\forall m, p \in \mathbb{N}$, $p \leq m$. From here, it results that $l_m l_{m+p} = l_{2m+p} + (-1)^m \cdot l_p$, $\forall m, p \in \mathbb{N}$. \square

3. Generalized Fibonacci-Lucas numbers and generalized Fibonacci-Lucas quaternions

Let n be an arbitrary positive integer and p, q be two arbitrary integers. The sequence g_n ($n \geq 1$), where

$$g_{n+1} = p f_n + q l_{n+1}, \quad n \geq 0$$

is called the *generalized Fibonacci-Lucas numbers*.

To emphasize the integer p and q , in the following we will use the notation $g_n^{p,q}$ instead of g_n .

Let $\mathbb{H}(\alpha, \beta)$ be the generalized real quaternion algebra, the algebra of the elements of the form $a = a_1 \cdot 1 + a_2 i + a_3 j + a_4 k$, where $a_i \in \mathbb{R}$, $i^2 = \alpha$, $j^2 = \beta$, $k = ij = -ji$. We denote by $\mathbf{t}(a)$ and $\mathbf{n}(a)$ the trace and the norm of a real quaternion a . The norm of a generalized quaternion has the following expression $\mathbf{n}(a) = a_1^2 - \alpha a_2^2 - \beta a_3^2 + \alpha \beta a_4^2$ and $\mathbf{t}(a) = 2a_1$. It is known that for $a \in \mathbb{H}(\alpha, \beta)$, we have $a^2 - \mathbf{t}(a)a + \mathbf{n}(a) = 0$. The quaternion algebra $\mathbb{H}(\alpha, \beta)$ is a *division algebra* if for all $a \in \mathbb{H}(\alpha, \beta)$, $a \neq 0$ we have $\mathbf{n}(a) \neq 0$, otherwise $\mathbb{H}(\alpha, \beta)$ is called a *split algebra*.

Let $\mathbb{H}_{\mathbb{Q}}(\alpha, \beta)$ be the generalized quaternion algebra over the rational field. We define the n -th *generalized Fibonacci-Lucas quaternion* to be the element of the form

$$G_n^{p,q} = g_n^{p,q} \cdot 1 + g_{n+1}^{p,q} \cdot i + g_{n+2}^{p,q} \cdot j + g_{n+3}^{p,q} \cdot k,$$

where $i^2 = \alpha$, $j^2 = \beta$, $k = ij = -ji$.

In the following proposition, for $\alpha = -1$ and $\beta = p$, we compute the norm for the n -th generalized Fibonacci-Lucas quaternions.

Proposition 3.1. *Let n, p be two arbitrary positive integers and q be an arbitrary integer. Let $G_n^{p,q}$ be the n -th generalized Fibonacci-Lucas quaternion. Then the norm of the element $G_n^{p,q}$ in the quaternion algebra $\mathbb{H}_{\mathbb{Q}}(-1, p)$ has the form*

i)

$$\mathbf{n}(G_n^{p,q}) = (p^3 + p^2 + 8p^2q + 15pq^2 + 2pq) f_{2n-1} + (-3p^3 + 5q^2 - 22p^2q - 40pq^2 + 2pq) f_{2n+1};$$

or

ii)

$$\mathbf{n}(G_n^{p,q}) = g_{2n}^{a,b},$$

where $a = 4p^3 + p^2 + 30p^2q + 55pq^2 - 5q^2$ and $b = -3p^3 + 5q^2 - 22p^2q - 40pq^2 + 2pq$.

Proof. i)

$$\begin{aligned} \mathbf{n}(G_n^{p,q}) &= G_n^{p,q} \overline{G_n^{p,q}} = (g_n^{p,q})^2 + (g_{n+1}^{p,q})^2 - p(g_{n+2}^{p,q})^2 - p(g_{n+3}^{p,q})^2 = \\ &= (pf_{n-1} + ql_n)^2 + (pf_n + ql_{n+1})^2 - p(pf_{n+1} + ql_{n+2})^2 - p(pf_{n+2} + ql_{n+3})^2 = \\ &= p^2(f_{n-1}^2 + f_n^2) - p^3(f_{n+1}^2 + f_{n+2}^2) + q^2(l_n^2 + l_{n+1}^2) - pq^2(l_{n+2}^2 + l_{n+3}^2) + \\ &\quad + 2pq(f_{n-1}l_n + f_nl_{n+1}) - 2p^2q(f_{n+1}l_{n+2} + f_{n+2}l_{n+3}) \end{aligned} \quad (3.1).$$

Using Proposition 2.1 (i; vi;viii; ix) and the relation (3.1), we obtain

$$\begin{aligned} \mathbf{n}(G_n^{p,q}) &= p^2 f_{2n-1} + 5q^2 f_{2n+1} + 2pq \left[f_{2n-1} + (-1)^n + f_{2n+1} + (-1)^{n+1} \right] - \\ &\quad - p^3 f_{2n+3} - 5pq^2 f_{2n+5} - 2p^2q \left[f_{2n+3} + (-1)^{n+2} + f_{2n+5} + (-1)^{n+3} \right] = \\ &= p^2 f_{2n-1} + 5q^2 f_{2n+1} - p^3 f_{2n+3} - 5pq^2 f_{2n+5} + 2pql_{2n} - 2p^2ql_{2n+4}. \end{aligned}$$

From Fibonacci recurrence, we obtain

$$f_{2n+3} = 3f_{2n+1} - f_{2n-1}, f_{2n+5} = 8f_{2n+1} - 3f_{2n-1}.$$

Using these equalities, Proposition 2.1 (vi) and Fibonacci recurrence, we obtain $l_{2n} = f_{2n+1} + f_{2n-1}$ and $l_{2n+4} = f_{2n+3} + f_{2n+5} = 11f_{2n+1} - 4f_{2n-1}$.

It results

$$\begin{aligned} \mathbf{n}(G_n^{p,q}) &= (p^3 + p^2 + 8p^2q + 15pq^2 + 2pq) f_{2n-1} + \\ &\quad + (-3p^3 + 5q^2 - 22p^2q - 40pq^2 + 2pq) f_{2n+1}. \end{aligned}$$

ii) Using Proposition 2.1 (vi), last equality becomes

$$\begin{aligned} \mathbf{n}(G_n^{p,q}) &= (4p^3 + p^2 + 30p^2q + 55pq^2 - 5q^2) f_{2n-1} + \\ &\quad + (-3p^3 + 5q^2 - 22p^2q - 40pq^2 + 2pq) l_{2n}. \end{aligned}$$

If we denote $a = 4p^3 + p^2 + 30p^2q + 55pq^2 - 5q^2$ and $b = -3p^3 + 5q^2 - 22p^2q - 40pq^2 + 2pq$, the last equality is equivalent with $\mathbf{n}(G_n^{p,q}) = af_{2n-1} + bl_{2n} = g_{2n}^{a,b}$. \square

Let A be a Noetherian integral domain with the field of the fractions K and let $\mathbb{H}_K(\alpha, \beta)$ be the generalized quaternion algebra. We recall that \mathcal{O} is an order in $\mathbb{H}_K(\alpha, \beta)$ if $\mathcal{O} \subseteq \mathbb{H}_K(\alpha, \beta)$ is an A -lattice of $\mathbb{H}_K(\alpha, \beta)$ (i.e a finitely generated A submodule of $\mathbb{H}_K(\alpha, \beta)$) which is also a subring of $\mathbb{H}_K(\alpha, \beta)$ (see [Vo; 10]).

In the following, we will built an order of a quaternion algebras using the generalized Fibonacci-Lucas quaternions. Also we will prove that Fibonacci-Lucas quaternions can have an algebra structure over \mathbb{Q} . For this, we make the following remarks.

Remark 3.1. *Let n be an arbitrary positive integer and p, q be two arbitrary integers. Let $(g_n^{p,q})_{n \geq 1}$ be the generalized Fibonacci-Lucas numbers. Then*

$$pf_{n+1} + ql_n = g_n^{p,q} + g_{n+1}^{p,0}, \forall n \in \mathbb{N}^*.$$

Proof.

$$pf_{n+1} + ql_n = pf_{n-1} + ql_n + pf_n = g_n^{p,q} + pf_n = g_n^{p,q} + g_{n+1}^{p,0}.$$

\square

Remark 3.2. *Let n be an arbitrary positive integer and p, q be two arbitrary integers. Let $(g_n^{p,q})_{n \geq 1}$ be the generalized Fibonacci-Lucas numbers and $(G_n^{p,q})_{n \geq 1}$ be the generalized Fibonacci-Lucas quaternion elements. Then:*

$$G_n^{p,q} = 0 \text{ if and only if } p = q = 0.$$

Proof. " \Leftarrow " It is trivial.

" \Rightarrow " If $G_n^{p,q} = 0$, since $\{1, i, j, k\}$ is a basis in $\mathbb{H}_{\mathbb{Q}}(\alpha, \beta)$, we obtain that $g_n^{p,q} = g_{n+1}^{p,q} = g_{n+2}^{p,q} = g_{n+3}^{p,q} = 0$. It results $g_{n-1}^{p,q} = g_{n+1}^{p,q} - g_n^{p,q} = 0$, ..., $g_2^{p,q} = 0$, $g_1^{p,q} = 0$. But $g_1^{p,q} = pf_0 + ql_1 = 2q$, therefore $q = 0$. From $g_2^{p,q} = 0$, we obtain $p = 0$. \square

Theorem 3.1. *Let M be the set*

$$M = \left\{ \sum_{i=1}^n 5G_{n_i}^{p_i, q_i} \mid n \in \mathbb{N}^*, p_i, q_i \in \mathbb{Z}, (\forall i) i = \overline{1, n} \right\} \cup \{1\}.$$

- 1) *The set M has a ring structure with quaternions addition and multiplication.*
- 2) *The set M is an order of the quaternion algebra $\mathbb{H}_{\mathbb{Q}}(\alpha, \beta)$.*
- 3) *The set $M' = \left\{ \sum_{i=1}^n 5G_{n_i}^{p'_i, q'_i} \mid n \in \mathbb{N}^*, p'_i, q'_i \in \mathbb{Q}, (\forall i) i = \overline{1, n} \right\} \cup \{1\}$ is a \mathbb{Q} -algebra.*

Proof. 2) First, we remark that $0 \in M$ (using Remark 3.2).

Now we prove that M is a \mathbb{Z} -submodule of $\mathbb{H}_{\mathbb{Q}}(\alpha, \beta)$.

Let $n, m \in \mathbb{N}^*$, $a, b, p, q, p', q' \in \mathbb{Z}$. It is easy to prove that

$$ag_n^{p,q} + bg_m^{p',q'} = g_n^{ap, aq} + g_m^{bp', bq'}.$$

This implies that

$$aG_n^{p,q} + bG_m^{p',q'} = G_n^{ap,aq} + G_m^{bp',bq'}.$$

From here, we get immediately that M is a \mathbb{Z} -submodule of the quaternion algebra $\mathbb{H}_{\mathbb{Q}}(\alpha, \beta)$. Since $\{1, i, j, k\}$ is a basis for this submodule, it results that M is a free \mathbb{Z} -module of rank 4.

Now, we prove that M is a subring of $\mathbb{H}_{\mathbb{Q}}(\alpha, \beta)$. It is enough to show that $5G_n^{p,q} \cdot 5G_m^{p',q'} \in M$. For this, if $m < n$, we calculate

$$\begin{aligned} 5g_n^{p,q} \cdot 5g_m^{p',q'} &= 5(pf_{n-1} + ql_n) \cdot 5(p'f_{m-1} + q'l_m) = \\ &= 25pp'f_{n-1}f_{m-1} + 25pq'f_{n-1}l_m + 25p'qf_{m-1}l_n + 25qq'l_nl_m \end{aligned} \quad (3.2).$$

Using Proposition 2.1 (ix, x, xi), Proposition 2.2, Remark 3.1 and the equality (3.2) we obtain:

$$\begin{aligned} 5g_n^{p,q} \cdot 5g_m^{p',q'} &= 5pp'[l_{m+n-2} + (-1)^m \cdot l_{n-m}] + 25pq'[f_{m+n-1} + (-1)^m \cdot f_{n-m-1}] + \\ &\quad + 25p'q[f_{m+n-1} + (-1)^m \cdot f_{n-m+1}] + 25qq'[l_{m+n} + (-1)^m \cdot l_{n-m}] = \\ &= 5(p'p'l_{m+n-2} + 5p'qf_{m+n-1}) + 5[5p'q(-1)^m \cdot f_{n-m+1} + pp'(-1)^m \cdot l_{n-m}] + \\ &\quad + 25(pq'f_{m+n-1} + qq'l_{m+n}) + 25[pq' \cdot (-1)^m \cdot f_{n-m-1} + qq' \cdot (-1)^m \cdot l_{n-m}] = \\ &= 5g_{m+n-2}^{5p'q, pp'} + 5g_{m+n-1}^{5p'q, 0} + 5g_{n-m}^{5p'q \cdot (-1)^m, pp' \cdot (-1)^m} + 5g_{n-m+1}^{5p'q \cdot (-1)^m, 0} + \\ &\quad + 5g_{m+n}^{5pq', 5qq'} + 5g_{n-m}^{5pq' \cdot (-1)^m, 5qq' \cdot (-1)^m}. \end{aligned}$$

Therefore, we obtain that $5G_n^{p,q} \cdot 5G_m^{p',q'} \in M$.

From here, it results that M is an order of the quaternion algebra $\mathbb{H}_{\mathbb{Q}}(\alpha, \beta)$.

1) and 3) are obviously. \square

Remark 3.3. For $\alpha = \beta = -1$, we have that M is included in the set of Hurwitz quaternions,

$$\mathcal{H} = \{q = a_1 + a_2i + a_3j + a_4k \in \mathbb{H}_{\mathbb{Q}}(-1, -1), a_1, a_2, a_3, a_4 \in \mathbb{Z} \text{ or } \mathbb{Z} + \frac{1}{2}\},$$

which is a maximal order in $\mathbb{H}_{\mathbb{Q}}(-1, -1)$.

From [Lam; 04], [Sa; 14], it is known that if p is an odd prime positive integer, the algebra $\mathbb{H}_{\mathbb{Q}}(-1, p)$ is a split algebra if and only if $p \equiv 1 \pmod{4}$. In the following, we will show that there are an infinite numbers of generalized Fibonacci-Lucas quaternion elements which are invertible in this algebra.

Proposition 3.2. *Let n be an arbitrary positive integer. Let $(f_n)_{n \geq 0}$ be the Fibonacci sequence and $(l_n)_{n \geq 0}$ be the Lucas sequence. Let p be an odd*

prime positive integer, $p \equiv 1 \pmod{4}$, q be an arbitrary integer. Let $G_n^{p,q}$ be the n -th generalized Fibonacci-Lucas quaternion and $\mathbb{H}_{\mathbb{Q}}(-1, p)$ be the quaternion algebra. The following statements are true:

- i) $\mathbf{n}(G_n^{p,q}) \neq 0$, $(\forall) (n, q) \in \mathbb{N}^* \times \mathbb{N}$;
- ii) If $p = 5$, then $\mathbf{n}(G_n^{5,q}) \neq 0$, $(\forall) (n, q) \in \mathbb{N}^* \times \mathbb{Z}_-$, $(n, q) \neq (1, -2)$;
- iii) if $p > 5$ and $n \equiv 0 \pmod{3}$, then $\mathbf{n}(G_n^{p,q}) \neq 0$, $(\forall) q \in \mathbb{Z}$.

Proof. We know that an element $x \neq 0$ from a quaternion algebra is invertible in this algebra if the norm $\mathbf{n}(x) \neq 0$.

i) From Proposition 3.1, we know that

$$\begin{aligned} \mathbf{n}(G_n^{p,q}) &= (p^3 + p^2 + 8p^2q + 15pq^2 + 2pq) f_{2n-1} + \\ &\quad + (-3p^3 + 5q^2 - 22p^2q - 40pq^2 + 2pq) f_{2n+1} \end{aligned} \quad (3.3).$$

If $q \in \mathbb{N}$, since $p \in \mathbb{N}^*$, it results

$$p^3 + p^2 + 8p^2q + 15pq^2 + 2pq < 3p^3 - 5q^2 + 22p^2q + 40pq^2 - 2pq.$$

Using the inequality $f_{2n-1} < f_{2n+1}$, we obtain that $\mathbf{n}(G_n^{p,q}) < 0$. So $\mathbf{n}(G_n^{p,q}) \neq 0$.

In the case when $q \in \mathbb{Z}_-$, if there exist generalized Fibonacci-Lucas quaternion elements with $\mathbf{n}(G_n^{p,q}) = 0$, using relation (3.3), we obtain that

$$\begin{aligned} p[(p^2 + p + 8pq + 15q^2 + 2q) f_{2n-1} + (-3p^2 - 22pq - 40q^2 + 2q) f_{2n+1}] &= \\ &= -5q^2 f_{2n+1}, \end{aligned} \quad (3.4).$$

therefore, $p \mid 5q^2 f_{2n+1}$. But we know from the hypothesis that p is a prime positive integer, $p \equiv 1 \pmod{4}$, so we consider two cases:

ii) The case $p = 5$. Using relation (3.3) and making some calculations we get that

$$\mathbf{n}(G_n^{5,q}) = 0 \Leftrightarrow (5q^2 + 10q + 30) f_{2n-1} - (39q^2 + 108q + 75) f_{2n+1} = 0 \quad (3.5).$$

Therefore

$$0 < 5q^2 + 10q + 30 < 39q^2 + 108q + 75, \quad (\forall) q \in \mathbb{Z}, q \neq -2; -1.$$

Using the fact that $f_{2n-1} < f_{2n+1}$, $(\forall) n \in \mathbb{N}^*$, it results

$$\mathbf{n}(G_n^{5,q}) < 0 \quad (\forall) n \in \mathbb{N}^*, (\forall) q \in \mathbb{Z}_-, q \neq -2; -1.$$

In the following, we will find if the equations $\mathbf{n}(G_n^{5,-1}) = 0$ respectively $\mathbf{n}(G_n^{5,-2}) = 0$ have solutions for $n \in \mathbb{N}^*$.

Using the relation (3.5) and the Fibonacci recurrence, we have

$$\mathbf{n}(G_n^{5,-1}) = 0 \Leftrightarrow 25f_{2n-1} - 6f_{2n+1} = 0 \Leftrightarrow 7f_{2n-1} + 6f_{2n-3} = 0.$$

Since $f_{2n-1}, f_{2n-3} > 0$ it results that does not exist $n \in \mathbb{N}^*$ such that $7f_{2n-1} + 6f_{2n-3} = 0$.

Using relation (3.5) and the Fibonacci recurrence, it results

$$\mathbf{n}(G_n^{5,-2}) = 0 \Leftrightarrow 30f_{2n-1} - 15f_{2n+1} = 0 \Leftrightarrow f_{2n-2} = 0 \Leftrightarrow n = 1.$$

iii) when $p > 5$, since $n \equiv 0 \pmod{3}$ it results that $2n - 1 \equiv 2 \pmod{3}$ and $2n + 1 \equiv 1 \pmod{3}$. Applying Proposition 2.1 (xiii), we obtain that f_{2n-1} and f_{2n+1} are odd numbers. Since p is an odd number, we obtain that the equality (3.4) cannot be held (is easy to prove that the equality (3.4) holds only when f_{2n-1} is an even number or f_{2n+1} is an even number). \square

For $a \in \mathbb{H}_{\mathbb{Q}}(\alpha, \beta)$. The centralizer of the element a is

$$C(a) = \{x \in \mathbb{H}_{\mathbb{Q}}(\alpha, \beta) \mid ax = xa\}.$$

From [Fl; 01], Proposition 2.13, we know that the equation

$$ax = bx, a, b \in \mathbb{H}_K(\alpha, \beta), \quad (3.6.)$$

where K is an arbitrary field of $\text{char} K \neq 0$, $a, b \notin K$, $a \neq \bar{b}$, has the solutions of the form

$$x = \lambda_1 (a - \mathbf{t}(a) + b - \mathbf{t}(b)) + \lambda_2 (\mathbf{n}(a - \mathbf{t}(a)) - (a - \mathbf{t}(a))(b - \mathbf{t}(b))), \lambda_1, \lambda_2 \in K \quad (3.7.)$$

if $\mathbb{H}_K(\alpha, \beta)$ is a division quaternion algebra or if $\mathbb{H}_K(\alpha, \beta)$ is a split quaternion algebra and $\mathbf{n}(a) \neq 0, \mathbf{n}(b) \neq 0$.

Proposition 3.3. *Let n be a positive integer, $n \equiv 0 \pmod{3}$. Let $(f_n)_{n \geq 0}$ be the Fibonacci sequence and $(l_n)_{n \geq 0}$ be the Lucas sequence. Let p be an odd prime positive integer, $p \equiv 1 \pmod{4}$, q be an arbitrary integer. Therefore, the centralizer of the element $G_n^{p,q} \in \mathbb{H}_{\mathbb{Q}}(-1, p)$ with $p > 5$, q arbitrary integer is the set*

$$C(G_n^{p,q}) = \{G_n^{\gamma, \delta} + \Lambda, \Lambda \in \mathbb{Q}\},$$

where $\gamma = 2\lambda_1 p, \delta = 2\lambda_1 q, \Lambda = g_n^{-2\lambda_1 p, -2\lambda_1 q} + g_{2n}^{2\lambda_2 a - 4\lambda_2 p q, 2\lambda_2 b - 2\lambda_2 q^2} - 2\lambda_2 \frac{p^2}{5} l_{2n-2} - 2\lambda_2 A$, with $\lambda_1, \lambda_2 \in \mathbb{Q}$ and $A = \frac{2p^2}{5}(-1)^n + 2q^2(-1)^n + 2pq(-1)^n$.

Proof. Since $C(G_n^{p,q}) = \{x \in \mathbb{H}_{\mathbb{Q}}(-1, p) \mid G_n^{p,q}x = xG_n^{p,q}\}$, using relations (3.6) and (3.7) for $a = b$, we obtain that the equation $G_n^{p,q}x = xG_n^{p,q}$ has the solutions of the form

$$x = 2\lambda_1 (G_n^{p,q} - \mathbf{t}(G_n^{p,q})) + 2\lambda_2 \mathbf{n}(G_n^{p,q} - \mathbf{t}(G_n^{p,q})), \lambda_1, \lambda_2 \in \mathbb{Q}. \quad (3.8.)$$

For $G_n^{p,q} = g_n^{p,q} \cdot 1 + g_{n+1}^{p,q} \cdot i + g_{n+2}^{p,q} \cdot j + g_{n+3}^{p,q} \cdot k$, we have $\mathbf{t}(G_n^{p,q}) = g_n^{p,q}$ and $\mathbf{n}(G_n^{p,q}) = g_{2n}^{a,b}$, with a and b as in Proposition 3.1. From here, we have that $\mathbf{n}(G_n^{p,q} - \mathbf{t}(G_n^{p,q})) = g_{2n}^{a,b} - g_n^{p,q} = g_{2n}^{a,b} - (pf_{n-1} + ql_n)^2$. Using Proposition 2.1, relations v), ix) and xii), it results

$$\mathbf{n}(G_n^{p,q} - \mathbf{t}(G_n^{p,q})) = g_{2n}^{a,b} - \frac{p^2}{5}(l_{2n-2}(-1)^n l_0) - q^2(l_{2n} + 2(-1)^n) - 2pq(f_{2n-1} + (-1)^n) =$$

$$\begin{aligned}
&= g_{2n}^{a,b} - \frac{p^2}{5} l_{2n-2} - \frac{2p^2}{5} (-1)^n - q^2 l_{2n} - 2q^2 (-1)^n - 2pq f_{2n-1} - 2pq (-1)^n = \\
&= g_{2n}^{a,b} + g_{2n}^{-2pq, -q^2} - \frac{p^2}{5} l_{2n-2} - \frac{2p^2}{5} (-1)^n - 2q^2 (-1)^n - 2pq (-1)^n = \\
&= g_{2n}^{a-2pq, b-q^2} - \frac{p^2}{5} l_{2n-2} - \frac{2p^2}{5} (-1)^n - 2q^2 (-1)^n - 2pq (-1)^n = \\
&= g_{2n}^{a-2pq, b-q^2} - \frac{p^2}{5} l_{2n-2} - A, \text{ where } A = \frac{2p^2}{5} (-1)^n + 2q^2 (-1)^n + 2pq (-1)^n. \text{ Using} \\
&\text{relation (3.8), we obtain } x = 2\lambda_1 (G_n^{p,q} - g_n^{p,q}) + 2\lambda_2 \left(g_{2n}^{a-2pq, b-q^2} - \frac{p^2}{5} l_{2n-2} - A \right) = \\
&= G_n^{2\lambda_1 p, 2\lambda_1 q} + g_n^{-2\lambda_1 p, -2\lambda_1 q} + g_{2n}^{2\lambda_2 a - 4\lambda_2 pq, 2\lambda_2 b - 2\lambda_2 q^2} - 2\lambda_2 \frac{p^2}{5} l_{2n-2} - 2\lambda_2 A, \lambda_1, \lambda_2 \in \mathbb{Q}. \square
\end{aligned}$$

Conclusions. In this paper, starting from a special set of elements, namely Fibonacci-Lucas quaternions, we proved that this set is an order –in the sense of ring theory– of the quaternion algebra $\mathbb{H}_{\mathbb{Q}}(\alpha, \beta)$ and it can have an algebra structure over \mathbb{Q} . It will be very interesting to find all properties of this algebra and to find conditions under which it is a division algebra or a split algebra.

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